

# A GODEL MODAL LOGIC <sup>1</sup>

(revised Dec. 2008)

Xavier Caicedo \*

Ricardo Oscar Rodríguez \*\*

\* Departamento de Matemáticas, Universidad de los Andes, Bogotá, Colombia  
xcaicedo@uniandes.edu.co

\*\* Departamento de Computación, Fac. Ciencias Exactas y Naturales  
Universidad de Buenos Aires, 1428 Buenos Aires, Argentina  
ricardo@dc.uba.ar

## 1 Introduction

Sometimes it is needed in approximate reasoning to deal simultaneously with both fuzziness of propositions and modalities, for instance one may try to assign a degree of truth to propositions like “John is possibly tall” or “John is necessarily tall”, where “John is tall” is presented as a fuzzy proposition. Fuzzy logic should be a suitable tool to model not only vagueness but also other kinds of information features like certainty, belief or similarity, which have a natural interpretation in terms of modalities.

We address in this paper the case of pure modal operators for Gödel logic, one of the main systems of fuzzy logic arising from Hájek [11] classification. For this purpose we consider a many-valued version of Kripke semantics for modal logic where both, propositions at each world and the accessibility relation, are infinitely valued in the standard Gödel algebra  $[0,1]$ .

We provide strongly complete axiomatizations for the  $\Box$ -fragment and the  $\Diamond$ -fragment of the resulting minimal logic. These fragments are shown to behave quite asymmetrically. Validity in the first one is univocally determined by the class of frames having a crisp (that is, two-valued) accessibility relation, while validity in the second requires truly fuzzy frames. In addition, the  $\Box$ -fragment does not enjoy the finite model property with respect to the number of worlds or the number of truth values while the  $\Diamond$ -fragment does.

---

<sup>1</sup>The results of this paper were announced at the meeting on “Logic, Computability and Randomness”, Córdoba, Argentina, Sept. 2004. Publication was delayed, aiming to axiomatize the full logic with both modal operators, which resulted elusive. Since the results have been quoted in some publications based in an incomplete preliminary manuscript, we have chosen to circulate this revision. We obtained recently the strong completeness of the full logic, result which will appear elsewhere.

We consider also the Gödel analogues of the classical modal systems T, S4 and S5 for each modal operator and show that the first two are characterized by the many-valued versions of the frame properties which characterize their classical counterparts.

Our approach is related to Fitting [7] who considers Kripke models taking values in a fixed finite Heyting algebra; however, his systems and completeness proofs depend essentially on finiteness of the algebra and the fact that his languages contain constants for all the truth values of the algebra. We most rely on completely different methods.

Modal logics with an intuitionistic basis and Kripke style semantics have been investigated in a number of relevant papers (see Ono [12], Fischer Servi [5], Bošzic and Došen [3], Font [8], Wolter [13], from an extensive literature), but in all cases the models carry two (or more) crisp accessibility relations satisfying some commuting properties: a pre-order to account for the intuitionistic connectives and one or more binary relations to account for the modal operators. Our semantics has, instead, a single arbitrary fuzzy accessibility relation and does not seem reducible to those multi-relational semantics since the latter enjoy the finite model property for  $\Box$  (cf. Grefe [10]).

We assume the reader is acquainted with modal and Gödel logics and the basic laws of linear Heyting algebras (cf. Chagrova [2]).

## 2 Gödel-Kripke models

The language  $\mathcal{L}_{\Box\Diamond}$  of propositional *Gödel modal logic* is built from a set  $Var$  of propositional variables, logical connectives symbols  $\wedge, \rightarrow, \perp$ , and the modal operator symbols  $\Box$  and  $\Diamond$ . Other connectives are defined:

$$\begin{aligned} \top &:= \varphi \rightarrow \varphi \\ \neg\varphi &:= \varphi \rightarrow \perp \\ \varphi \vee \psi &:= ((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightarrow \varphi) \\ \varphi \longleftrightarrow \psi &:= (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi). \end{aligned}$$

$\mathcal{L}_{\Box}$  and  $\mathcal{L}_{\Diamond}$  will denote, respectively, the  $\Box$ -fragment and the  $\Diamond$ -fragment of the language.

As stated before, the semantics of Gödel modal logic will be based in fuzzy Kripke models where the valuations at each world and also the accessibility relation between worlds are  $[0, 1]$ -valued. The symbols  $\cdot$  and  $\Rightarrow$  will

denote the Gödel norm in  $[0, 1]$  and its residuum, respectively:

$$a \cdot b = \min\{a, b\}, \quad a \Rightarrow b = \begin{cases} 1, & \text{if } a \leq b \\ b, & \text{otherwise} \end{cases}$$

the derived maximum and pseudo-complement operations will be denoted  $\vee$  and  $-$ , respectively. This yields the standard Gödel algebra; that is, the unique Heyting algebra structure in the linearly ordered interval.

**Definition 2.1** A *Gödel-Kripke model* (GK-model) will be a structure  $\langle W, S, e \rangle$  where:

- $W$  is a non-empty set of objects that we call *worlds* of  $M$ .
- $S : W \times W \rightarrow [0, 1]$  is an arbitrary function  $(x, y) \mapsto Sxy$ .
- $e : W \times Var \rightarrow [0, 1]$  is an arbitrary function  $(x, p) \mapsto e(x, p)$ .

The evaluations  $e(x, -) : Var \rightarrow [0, 1]$  are extended simultaneously to all formula in  $\mathcal{L}_{\Box\Diamond}$  by defining inductively at each world  $x$ :

$$\begin{aligned} e(x, \varphi \wedge \psi) &:= e(x, \varphi) \cdot e(x, \psi) \\ e(x, \varphi \rightarrow \psi) &:= e(x, \varphi) \Rightarrow e(x, \psi) \\ e(x, \perp) &:= 0 \\ e(x, \Box\varphi) &:= \inf_{y \in W} \{Sxy \Rightarrow e(y, \varphi)\} \\ e(x, \Diamond\varphi) &:= \sup_{y \in W} \{Sxy \cdot e(y, \varphi)\}. \end{aligned}$$

It follows that  $e(x, \varphi \vee \psi) = e(x, \varphi) \vee e(x, \psi)$  and  $e(x, \neg\varphi) = -e(x, \varphi)$ .

The notions of a formula  $\varphi$  being true at a world  $x$ , valid in a model  $M = \langle W, S, e \rangle$ , or universally valid, are the usual ones:

- $\varphi$  is *true in  $M$  at  $x$* , written  $M \models_x \varphi$ , iff  $e(x, \varphi) = 1$ .
- $\varphi$  is *valid in  $M$* , written  $M \models \varphi$ , iff  $M \models_x \varphi$  at any world  $x$  of  $M$ .
- $\varphi$  is *GK-valid*, written  $\models_{GK} \varphi$ , if it is valid in all the GK-models.

Clearly, all valid schemes of Gödel logic are GK-valid. In addition,

**Proposition 2.1** . *The following modal schemes are GK-valid:*

$$\begin{array}{ll} \mathbf{K}_{\Box} & \Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi) \\ \mathbf{Z}_{\Box} & \neg\neg\Box\theta \rightarrow \Box\neg\neg\theta \\ \mathbf{D}_{\Diamond} & \Diamond(\varphi \vee \psi) \rightarrow (\Diamond\varphi \vee \Diamond\psi) \quad (\text{in fact, an equivalence}) \\ \mathbf{Z}_{\Diamond} & \Diamond\neg\neg\varphi \rightarrow \neg\neg\Diamond\varphi \\ \mathbf{F}_{\Diamond} & \neg\Diamond\perp \end{array}$$

*Proof:* Let  $M = \langle W, S, e \rangle$  be an arbitrary Gödel-Kripke model and  $x \in W$ .

(**K** $_{\Box}$ ) By definition 2.1 and properties of the residuum we have for any  $y \in W$ :  $e(x, \Box(\varphi \rightarrow \psi)) \cdot e(x, \Box\varphi) \leq (Sxy \Rightarrow (e(y, \varphi) \Rightarrow e(y, \psi))) \cdot (Sxy \Rightarrow e(y, \varphi)) \leq (Sxy \Rightarrow e(y, \psi))$ . Taking the meet over  $y$  in the last expression:  $e(x, \Box(\varphi \rightarrow \psi)) \cdot e(x, \Box\varphi) \leq e(x, \Box\psi)$ , hence  $e(x, \Box(\varphi \rightarrow \psi)) \leq e(x, \Box\varphi \rightarrow \Box\psi)$ .

(**Z** $_{\Box}$ ) Utilizing the Heyting algebra identity:  $--(x \Rightarrow y) = (x \Rightarrow --y)$ , we have:  $e(x, \neg\neg\Box\theta) = --e(x, \Box\theta) \leq --(Sxy \Rightarrow e(y, \theta)) = (Sxy \Rightarrow --e(y, \theta)) = (Sxy \Rightarrow e(y, \neg\neg\theta))$ . Taking meet over  $y$  in the last expression:  $e(x, \neg\neg\Box\theta) \leq e(x, \Box\neg\neg\theta)$ .

(**D** $_{\Diamond}$ ) By properties of suprema and distributivity of  $\Upsilon$  over  $\cdot$ ,  $e(\Diamond(\varphi \vee \psi)) = \sup_y \{Sxy \cdot (e(y, \varphi) \Upsilon e(y, \psi))\} = \sup_y \{Sxy \cdot e(y, \varphi)\} \Upsilon \sup_y \{Sxy \cdot e(y, \psi)\}$

(**Z** $_{\Diamond}$ )  $Sxy \cdot e(\neg\neg\varphi, y) \leq --(Sxy \cdot e(\varphi, y)) \leq --e(\Diamond\varphi, x) = e(\neg\neg\Diamond\varphi, x)$ .

(**F** $_{\Diamond}$ )  $e(x, \Diamond\perp) = \sup_y \{Sxy \cdot 0\} = 0$ . ■

The Modus Ponens rule preserves truth at every world of any GK-model. On the other hand, the classical introduction rules for the modal operators

$$\mathbf{RN}_{\Box} : \frac{\varphi}{\Box\varphi} \qquad \mathbf{RN}_{\Diamond} : \frac{\varphi \rightarrow \psi}{\Diamond\varphi \rightarrow \Diamond\psi} .$$

do not preserve local truth. However,

**Proposition 2.2**  $\mathbf{RN}_{\Box}$  and  $\mathbf{RN}_{\Diamond}$  preserve validity at any given model, thus they preserve GK-validity.

*Proof:* . (**RN** $_{\Box}$ ) If  $e(x, \varphi) = 1$  for all  $x$  then  $e(x, \Box\varphi) = \inf_y \{Sxy \Rightarrow e(y, \varphi)\} = \inf\{1\} = 1$  for all  $x$ . (**RN** $_{\Diamond}$ ) If  $e(x, \varphi \rightarrow \psi) = 1$  for all  $x$  then  $Sxy \cdot e(y, \varphi) \leq Sxy \cdot e(y, \psi) \leq e(x, \Diamond\psi)$ . Taking join over  $y$  in the left hand side of the last inequality,  $e(x, \Diamond\varphi) \leq e(x, \Diamond\psi)$ . ■

Semantic consequence is defined for any theory  $T \subseteq \mathcal{L}_{\Box\Diamond}$ , as follows:

**Definition 2.2**  $T \models_{GK} \varphi$  if and only if for any GK-model  $\mathcal{M}$  and any world  $x$  in  $\mathcal{M}$ ,  $\mathcal{M} \models_x T$  implies  $\mathcal{M} \models_x \varphi$ .

An alternative notion of logical consequence arises naturally. Set  $e(x, T) = \{e(x, \varphi) : \varphi \in T\}$  then:

**Definition 2.3**  $T \models_{GK\leq} \varphi$  if and only if for any GK-model  $\mathcal{M}$  and any world  $x$  in  $\mathcal{M}$ ,  $\inf e(x, T) \leq e(x, \varphi)$ .

Clearly,  $\models_{GK\leq}$  implies  $\models_{GK}$ , and it will follow from our completeness theorems that both notions are equivalent for countable theories. This fact has been already observed for pure Gödel logic by Baaz and Zach in [1].

Note that Modus Ponens preserves consequence but this is not the case of the inference rules  $\mathbf{RN}_{\Box}$  and  $\mathbf{RN}_{\Diamond}$ .

### 3 On strong completeness of Gödel logic

To prove strong completeness of the unimodal fragments  $\mathcal{L}_{\Box}$  and  $\mathcal{L}_{\Diamond}$  we will reduce the problem to pure Gödel propositional logic.

In the rest of this paper  $\mathcal{L}(X)$  will denote the Gödel language built from a set of propositional variables  $X$  and the connectives  $\wedge, \rightarrow, \perp$ .

Let  $\mathcal{G}$  be a fixed axiomatic calculus for Gödel logic, say the following one given by Hájek ([11], Def. 4.2.3.):

$$\begin{aligned} &(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi)) \\ &(\varphi \wedge \psi) \rightarrow \varphi \\ &(\varphi \wedge \psi) \rightarrow (\psi \wedge \varphi) \\ &(\varphi \rightarrow (\psi \rightarrow \chi)) \longleftrightarrow ((\varphi \wedge \psi) \rightarrow \chi) \\ &((\varphi \wedge \psi) \rightarrow \chi) \longleftrightarrow (\varphi \rightarrow (\psi \rightarrow \chi)) \\ &\varphi \rightarrow (\varphi \wedge \varphi) \\ &((\varphi \rightarrow \psi) \rightarrow \chi) \rightarrow (((\psi \rightarrow \varphi) \rightarrow \chi) \rightarrow \chi) \\ &\perp \rightarrow \varphi \end{aligned}$$

**MP:** From  $\varphi$  and  $\varphi \rightarrow \psi$ , infer  $\psi$

$\vdash$  will denote deduction in this calculus.

It is well known that  $\mathcal{G}$  is deductively equivalent to Dummett logic, the intermediate logic obtained by adding to Heyting calculus the pre-linearity schema:

$$(\varphi \rightarrow \psi) \vee (\psi \rightarrow \varphi).$$

Given a valuation  $v : X \rightarrow [0, 1]$ , let  $\bar{v}$  denote the extension of  $v$  to  $\mathcal{L}(X)$  according to the Gödel interpretation of the connectives. We will need the following strong form of standard completeness for Gödel logic:

**Proposition 3.1** *Let  $T$  be a countable theory and  $U$  a countable set of formulas of  $\mathcal{L}(X)$  such that for every finite  $S \subseteq U$  we have  $T \not\vdash \bigvee S$  then there is a valuation  $v : X \rightarrow [0, 1]$  such that  $\bar{v}(\alpha) = 1$  for all  $\alpha \in T$  and  $\bar{v}(\beta) < 1$  for each  $\beta \in U$ .*

*Proof:* Extend  $T$  to a prime theory  $T'$  (that is,  $T' \vdash \alpha \vee \beta$  implies  $T' \vdash \alpha$  or  $T' \vdash \beta$ ) satisfying the same hypothesis with respect to  $U$  (this is standard). The Lindenbaum algebra  $\mathcal{L}(X)/_{\equiv T'}$  of  $T'$  is linearly ordered since by primality and the pre-linearity schema  $T' \vdash \alpha \rightarrow \beta$  or  $T' \vdash \beta \rightarrow \alpha$ . Moreover, the valuation  $v : X \rightarrow \mathcal{L}(X)/_{\equiv T'}$ ,  $v(x) = x/_{\equiv T'}$  is such that  $v(T) = 1$ ,  $v(\beta) < 1$  for all  $\beta \in U$ . As  $T'$  is countable we may assume  $X$  is countable and thus, being also countable,  $\mathcal{L}(X)/_{\equiv T'}$  is embeddable in the Gödel algebra  $[0, 1]$ , therefore, we may assume  $v : X \rightarrow [0, 1]$ . ■

From the proposition we obtain the usual formulation of completeness for countable  $T$ . We can not expect strong standard completeness of  $\mathcal{G}$  for uncountable theories, as the following example illustrates.

**Example.** Set  $T = \{(p_\beta \rightarrow p_\alpha) \rightarrow q : \alpha < \beta < \omega_1\}$  where  $\omega_1$  is the first uncountable cardinal, then  $T \not\vdash q$ . Otherwise we would have  $\Sigma \vdash q$ , for some finite  $\Sigma = \{(p_{\alpha_{i+1}} \rightarrow p_{\alpha_i}) \rightarrow q : 1 \leq i < n\}$ , but this is not possible by soundness of  $\mathcal{G}$ , because the valuation  $v(q) = \frac{1}{2}$ ,  $v(p_{\alpha_i}) = \frac{1}{2}(1 - \frac{1}{i+1})$  for  $1 \leq i \leq n$ , makes  $v(p_{\alpha_i}) < v(p_{\alpha_{i+1}}) < \frac{1}{2}$  and thus  $\bar{v}((p_{\alpha_{i+1}} \rightarrow p_{\alpha_i}) \rightarrow q) = 1$  for  $1 \leq i < n$ , while  $v(q) < 1$ . On the other hand, there is no valuation  $v$  such that  $\bar{v}(T) = 1$  and  $v(q) < 1$ , because that would imply  $\bar{v}(p_\beta \rightarrow p_\alpha) < 1$  for all  $\alpha < \beta < \omega_1$ , and thus the set  $\{v(p_\alpha) : \alpha < \omega_1\}$  would be ordered in type  $\omega_1$ , which is impossible because any well ordered subset of  $([0, 1], <)$  is at most countable.

## 4 Completeness of the $\Box$ -fragment

Let  $\mathcal{G}_\Box$  be the formal system on the language  $\mathcal{L}_\Box$  which is obtained by adding to the system  $\mathcal{G}$  for Gödel logic (applied to  $\mathcal{L}_\Box$ ) the following axiom schemes and rule:

**K $_\Box$ :**  $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$

**Z $_\Box$ :**  $\neg\neg\Box\theta \rightarrow \Box\neg\neg\theta$

**NR $_\Box$ :** From  $\varphi$  infer  $\Box\varphi$

$\vdash_{\mathcal{G}_\Box} \varphi$  expresses theorem-hood in this logic. Proofs with assumptions are also allowed, with the restriction that **NR $_\Box$**  is to be applied to theorems only (or, what amounts to the same, to previous steps of the proof not depending on the assumptions).  $T \vdash_{\mathcal{G}_\Box} \varphi$  will express that there is such a proof of  $\varphi$  with assumptions from the set  $T$ .

The deduction theorem follows readily by induction in the length of proofs:

**DT:**  $T \cup \{\alpha\} \vdash_{\mathcal{G}_\square} \varphi$  implies  $T \vdash_{\mathcal{G}_\square} \alpha \rightarrow \varphi$ .

Applying consecutively **DT**, **NR** $_\square$ , **K** $_\square$ , and **MP**, we obtain the derived rule:

**Lemma 4.1** *If  $\mu_1, \dots, \mu_k \vdash_{\mathcal{G}_\square} \varphi$  then  $\Box\mu_1, \dots, \Box\mu_k \vdash_{\mathcal{G}_\square} \Box\varphi$ .*

We obtain also soundness of  $\mathcal{G}_\square$ :

**Lemma 4.2**  *$T \vdash_{\mathcal{G}_\square} \varphi$  implies  $T \models_{GK\leq} \varphi$ , hence,  $T \models_{GK} \varphi$ .*

*Proof:* By the deduction theorem,  $T \vdash_{\mathcal{G}_\square} \varphi$  implies  $\vdash_{\mathcal{G}_\square} (\wedge\Sigma \rightarrow \varphi)$  for some finite  $\Sigma \subseteq T$ . Since the axioms of  $\mathcal{G}_\square$  are valid in all GK-models (Prop. 2.1) and **MP**, **NR** $_\square$ , **K** $_\square$  preserve validity (Prop. 2.2) then  $\models_{GK} (\wedge\Sigma \rightarrow \varphi)$ . Therefore,  $\inf e(x, T) \leq e(x, \wedge\Sigma) \leq e(x, \varphi)$  for any world  $x$  in any GK-model. ■

Let

$$T\mathcal{G}_\square = \{A : A \text{ is a theorem of } \mathcal{G}_\square\}$$

Since all uses of **NR** $_\square$  in a proof of  $T \vdash_{\mathcal{G}_\square} \varphi$  produce theorems of  $\mathcal{G}_\square$ , the proof may be seen as one in which Modus Ponens is the only rule utilized and  $T\mathcal{G}_\square$  is part of the assumptions. That is,

**Lemma 4.3** .  *$T \vdash_{\mathcal{G}_\square} \varphi$  if and only if  $T \cup T\mathcal{G}_\square \vdash \varphi$  in pure Gödel logic.*

To prove strong completeness of  $\mathcal{G}_\square$  we will define a canonical GK-model with the property that for any countable theory  $T$  and any formula  $\varphi$  such that  $T \not\vdash_{\mathcal{G}_\square} \varphi$ , there is a world  $x$  in the model which assigns the value 1 to  $T$  but less than 1 to  $\varphi$ . A surprising fact will be that this may be achieved with a model where the accessibility relation is crisp.

Let  $\Box\mathcal{L}_\square = \{\Box\theta : \theta \in \mathcal{L}_\square\}$  be the set of formulas in  $\mathcal{L}_\square$  which start with the connective  $\Box$ . Then any formula in  $\mathcal{L}_\square$  may be seen as a formula of the pure Gödel language built from  $X = Var \cup \Box\mathcal{L}_\square$  by means of  $\wedge, \neg, \perp$ . That is, we may consider the formulas in  $\Box\mathcal{L}_\square$  as additional propositional variables for Gödel logic.

**Canonical model**  $\mathcal{M}_\square = (W^*, S^*, e^*)$ .

- The set of worlds  $W^*$  will consist of those valuations  $v : Var \cup \Box\mathcal{L}_\square \rightarrow [0, 1]$  which satisfy  $\bar{v}(T\mathcal{G}_\square) = 1$  when extended to  $\bar{v} : \mathcal{L}_\square = \mathcal{L}(Var \cup \Box\mathcal{L}_\square) \rightarrow [0, 1]$  according to the Gödel interpretation of  $\wedge, \rightarrow, \perp$ .

- The fuzzy accessibility relation between worlds in  $\mathcal{M}_\square$  (actually a crisp relations) will be given by

$$S^*vw = \begin{cases} 1, & \text{if } v(\square\theta) \leq \overline{w}(\theta), \text{ for all } \theta \in \mathcal{L}_\square \\ 0, & \text{otherwise} \end{cases},$$

- The valuation associated to the world  $v$  will be  $v \upharpoonright Var$ . That is,  $e^*(p, v) = v(p)$  for any  $p \in Var$ .

For the sake of simplicity, we will write  $v(\varphi)$  for  $\overline{v}(\varphi)$ , from now on.

**Lemma 4.4** For any world  $v$  in the canonical model  $\mathcal{M}_\square$  and any  $\varphi$

$$e^*(v, \varphi) = v(\varphi).$$

*Proof:* This is proven by induction in the complexity of  $\varphi$  seen again as a formula of  $\mathcal{L}_\square$ . The atomic step and the inductive steps for the Gödel connectives being straightforward, it is enough to verify inductively  $e^*(v, \square\varphi) = v(\square\varphi)$ . By induction hypothesis we may assume  $e^*(w, \varphi) = w(\varphi)$  for any  $w$ , and thus we must show

$$v(\square\varphi) = \inf_w \{w(\varphi) : S^*vw = 1\}$$

By definition,  $S^*vw = 1$  implies  $v(\square\varphi) \leq w(\varphi)$ , hence

$$v(\square\varphi) \leq \inf_w \{w(\varphi) : S^*vw = 1\}.$$

Since equality above is trivial for  $v(\square\varphi) = 1$ , it remains only to show in case  $v(\square\varphi) = \alpha < 1$  that

$$\inf_w \{w(\varphi) : S^*vw = 1\} \leq \alpha. \quad (1)$$

That is, for any  $\epsilon > 0$  there is  $w$  such that  $S^*vw = 1$  and  $w(\varphi) < \alpha + \epsilon$ . To see this we prove first:

**Claim.** Let  $v$  be a world of  $\mathcal{M}_\square$  and  $\varphi$  be such that  $v(\square\varphi) = \alpha < 1$ , then there exists a world  $u$  of  $\mathcal{M}_\square$  such that  $u(\varphi) < 1$  and

- (i)  $u(\theta) = 1$  if  $v(\square\theta) > \alpha$
- (ii)  $u(\theta) > 0$  if  $v(\square\theta) > 0$ .



*Proof:* Assume  $v(\Box\varphi) = \alpha < 1$  and set

$$T_{\varphi,v} = \{\theta : v(\Box\theta) > \alpha\} \cup \{\neg\neg\theta : v(\Box\theta) > 0\}$$

Notice that  $v(\Box\mu) > \alpha$  for any  $\mu \in T_{\varphi,v}$  because  $v(\Box\theta) > 0$  implies  $v(\neg\neg\Box\theta) = 1$ , and thus  $v(\Box\neg\neg\theta) = 1$  since  $v$  satisfies axiom  $\mathbf{Z}_{\Box}$ . This implies that  $T_{\varphi,v} \not\vdash_{\mathcal{G}_{\Box}} \varphi$ . Otherwise,  $\mu_1, \dots, \mu_k \vdash_{\mathcal{G}_{\Box}} \varphi$  for some  $\mu_i \in T_{\varphi,v}$  and thus

$$\Box\mu_1, \dots, \Box\mu_k \vdash_{\mathcal{G}_{\Box}} \Box\varphi$$

by Lemma 4.1. Hence, by Lemma 4.2 and the previous observations,

$$\alpha < \min\{\Box\mu_1, \dots, \Box\mu_k\} \leq v(\Box\varphi),$$

a contradiction. By Lemma 4.3 we have  $T_{\varphi,v} \cup T\mathcal{G}_{\Box} \not\vdash \varphi$  and by countability of  $T_{\varphi,v} \cup T\mathcal{G}_{\Box}$  we may use the completeness theorem of Gödel logic (Proposition 3.1) to get a Gödel valuation  $u : L \rightarrow [0, 1]$  such that  $u(T_{\varphi,v}) = 1$  and  $u(\varphi) < 1$ . Then  $u \in \mathcal{M}_{\Box}$  and (i) holds by construction. Moreover, (ii) is satisfied because  $u(\neg\neg\theta) = 1$  and thus  $u(\theta) > 0$  if  $v(\Box\theta) > 0$ . This ends the proof of the claim.

Pick now an strictly increasing function  $g : [0, 1] \rightarrow [0, 1]$  such that

$$g(1) = 1, \quad g(0) = 0, \quad \text{and} \quad g((0, 1]) = (\alpha, \alpha + \epsilon).$$

As  $g$  is an homomorphism of Heyting algebras, the valuation  $w = g \circ u$  preserves the value 1 of the formulas in  $T\mathcal{G}_{\Box}$  and thus it belongs to  $\mathcal{M}_{\Box}$ . Moreover,  $v(\Box\theta) \leq w(\theta)$  for all  $\theta$ :

- if  $v(\Box\theta) > \alpha$  because  $w(\theta) = g(u(\theta)) = g(1) = 1$  by (1) above.
- if  $0 < v(\Box\theta) \leq \alpha$  because then  $0 < u(\theta) \leq 1$  by (2) above, and thus  $w(\theta) = g(u(\theta)) \in (\alpha, \alpha + \epsilon) \cup \{1\}$ .

This means  $S^*vw = 1$ , and since  $u(\varphi) < 1$  we have,  $w(\varphi) = g(u(\varphi)) < \alpha + \epsilon$ , which shows 1. ■

Call a GK-model *crisp* if  $S : W \times W \rightarrow \{0, 1\}$ , and write  $T \models_{Crisp} \varphi$  if the consequence relation holds at each node of any crisp GK-model.

**Theorem 4.1** *For any countable theory  $T$  and formula  $\varphi$  in  $\mathcal{L}_{\Box}$  the following are equivalent:*

- (i)  $T \vdash_{\mathcal{G}_{\Box}} \varphi$
- (ii)  $T \models_{GK_{\leq}} \varphi$
- (iii)  $T \models_{GK} \varphi$
- (iv)  $T \models_{Crisp} \varphi$ .

*Proof:* By Lemma 4.2, it is enough to show (iv)  $\Rightarrow$  (i). If  $T \not\models_{\mathcal{G}_\square} \varphi$  then  $T \cup T\mathcal{G}_\square \not\models \varphi$  by Lemma 4.3, and by strong completeness of Gödel logic there is a valuation  $v : Var \cup \square\mathcal{L}_\square \rightarrow [0, 1]$  such that  $\bar{v}(T) = \bar{v}(T\mathcal{G}_\square) = 1$  and  $\bar{v}(\varphi) < 1$ . Hence,  $v \in W^*$  by definition,  $e^*(v, T) = \bar{v}(T) = 1$ , and  $e^*(v, \varphi) = \bar{v}(\varphi) < 1$  by Lemma 4.4, showing that  $\mathcal{M}_\square \models_v T$  but  $\mathcal{M}_\square \not\models_v \varphi$ . That is,  $T \not\models_{Crisp} \varphi$  because the canonical model is crisp. ■

The example in Section 3 shows that standard strong completeness does not hold in modal Gödel logic with respect to uncountable theories.

## 5 $\mathcal{G}_\square$ does not have the finite model property

The following example shows that  $\mathcal{G}_\square$  does not have the finite model property with respect to GK-models. The scheme

$$\square \neg \neg \theta \rightarrow \neg \neg \square \theta,$$

reciprocal of axiom  $\mathbf{Z}_\square$ , is not valid because it fails in the (crisp) model  $\mathcal{M} = (\mathbb{N}, S, e)$ , where

$$\begin{aligned} Smn &= 1 \text{ for all } m, n \\ e(n, p) &= \frac{1}{n+1} \text{ for all } n \end{aligned}$$

Indeed,  $e(n, \neg \neg p) = - - \frac{1}{n+1} = 1$  for all  $n$  and thus,  $e(0, \square \neg \neg p) = \inf\{1 \Rightarrow 1\} = 1$ . On the other hand,  $e(0, \square p) = \inf_{n \in \mathbb{N}}\{1 \Rightarrow \frac{1}{n+1}\} = 0$ , and thus  $e(0, \neg \neg \square p) = 0$ .

However,

**Theorem 5.1**  $\square \neg \neg \theta \rightarrow \neg \neg \square \theta$  is valid in any GK-model  $\langle W, S, e \rangle$  with finite  $W$ .

*Proof:* Given a model  $\mathcal{M} = \langle W, S, e \rangle$ , we have:

$$e(v, \square \neg \neg \theta) = \begin{cases} 0, & \exists w \in W : Swv > 0 \text{ and } e(w, \theta) = 0 \\ 1, & \text{otherwise} \end{cases} \quad (2)$$

Now,  $e(v, \neg \neg \square \theta) = 0$  iff and only if  $e(v, \square \theta) = 0$ , which means there is a sequence of worlds  $\{w_n\}_n$  such that  $\{Sw_n \Rightarrow e(w_n, \theta)\}_n$  converges to 0, that is,

$$e(\neg \neg \square \theta, v) = \begin{cases} 0, & \text{if } \exists \{w_n\} \subseteq W : Sw_n > e(w_n, \theta) \text{ for all } n \in \mathbb{N}, \text{ and } \{e(w_n, \theta)\}_n \text{ converges to } 0 \\ 1, & \text{otherwise} \end{cases} \quad (3)$$

Assume  $e(v, \Box \neg \neg \theta) = 1$ . Then, according to (2),  $Svw = 0$  or  $e(w, \theta) > 0$  for any  $w \in W$ . If we had  $e(v, \neg \neg \Box \theta) = 0$ , then the sequence  $\{w_n\}_{n \in \mathbb{N}}$  given by (3) would satisfy:  $Svw_n > e(w_n, \theta)$ , and hence  $e(w_n, \theta) > 0$  for all  $n$  by the previous observation. If  $W$  is finite, the set  $\{e(w_n, \theta) : n \in \mathbb{N}\}$  has a minimum positive value and thus the sequence,  $\{e(w_n, \theta)\}_n$  would not converge to 0, a contradiction. ■

The proof of the theorem shows that  $\Box \neg \neg \theta \rightarrow \neg \neg \Box \theta$  would be valid in all GK models with finite Gödel algebra of values.

## 6 Completeness of the $\Diamond$ -fragment

The system  $\mathcal{G}_\Diamond$  results by adding to  $\mathcal{G}$  the following axiom schemes and rule in the language  $\mathcal{L}_\Diamond$ :

$$\begin{aligned} \mathbf{D}_\Diamond: & \Diamond(\varphi \vee \psi) \rightarrow (\Diamond\varphi \vee \Diamond\psi) \\ \mathbf{Z}_\Diamond: & \Diamond\neg\neg\varphi \rightarrow \neg\neg\Diamond\varphi \\ \mathbf{F}_\Diamond: & \neg\Diamond\perp \\ \mathbf{RN}_\Diamond: & \text{From } \varphi \rightarrow \psi \text{ infer } \Diamond\varphi \rightarrow \Diamond\psi \end{aligned}$$

As in the case of the  $\Box$ -fragment, in proofs with assumptions the rule  $\mathbf{RN}_\Diamond$  is to be used in theorems only, and under this definition we have the deduction theorem  $\mathbf{DT}$ , the derived rule:

**Lemma 6.1** *If  $\varphi \vdash_{\mathcal{G}_\Diamond} \psi$  then  $\Diamond\varphi \vdash_{\mathcal{G}_\Diamond} \Diamond\psi$ .*

and the soundness theorem:

**Lemma 6.2**  *$T \vdash_{\mathcal{G}_\Diamond} \varphi$  implies  $T \models_{GK \leq} \varphi$ , hence,  $T \models_{GK} \varphi$ .*

Let  $T\mathcal{G}_\Diamond$  be the set of theorem of  $\mathcal{G}_\Diamond$ , then it follows, as in the case of  $\mathcal{G}_\Box$ , that

**Lemma 6.3**  *$T \vdash_{\mathcal{G}_\Diamond} \varphi$  if and only if  $T \cup T\mathcal{G}_\Diamond \vdash \varphi$  in Gödel logic.*

Let  $\Diamond\mathcal{L}_\Diamond = \{\Diamond\theta : \theta \in \mathcal{L}_\Diamond\}$ . The canonical model  $\mathcal{M}_\Diamond = (W^*, S^*, e^*)$  is defined as follows:

- $W^*$  is the set of valuations  $v : Var \cup \Diamond\mathcal{L}_\Diamond \rightarrow [0, 1]$  such that  $v(T\mathcal{G}_\Diamond) = 1$  and its positive values have a positive lower bound:

$$\inf_{\varphi \in \mathcal{L}_\Diamond} \{v(\theta) : v(\theta) > 0\} = \delta > 0 \quad (4)$$

when the formulas in  $\Diamond \mathcal{L}_\Diamond$  are seen as propositional variables and  $v$  is extended to  $\mathcal{L}_\Diamond = \mathcal{L}(Var \cup \Diamond \mathcal{L}_\Diamond)$  as a Gödel valuation.

- The fuzzy relation between worlds in  $\mathcal{M}_\Diamond$  is given by

$$S^*vw := \inf_{\varphi \in \mathcal{L}_\Diamond} \{w(\theta) \Rightarrow v(\Diamond\theta)\}.$$

- $e^*(v, p) := v(p)$  for any  $p \in Var$ .

**Lemma 6.4** For any world  $v$  in the canonical model  $\mathcal{M}_\Diamond$  and any  $\varphi \in \mathcal{L}_\Diamond$  we have  $e^*(v, \varphi) = v(\varphi)$ .

*Proof:* The only non trivial step in a proof by induction on complexity of formulas of  $\mathcal{L}_\Diamond$  is that of  $\Diamond$ . By induction hypothesis,  $e^*(v, \Diamond\varphi) = \sup_w \{S^*vw \cdot e^*(w, \varphi)\} = \sup_w \{S^*vw \cdot w(\varphi)\}$ , then we must show  $\sup_w \{S^*vw \cdot w(\varphi)\} = v(\Diamond\varphi)$ . By definition

$$S^*vw \leq w(\varphi) \Rightarrow v(\Diamond\varphi),$$

for any  $\varphi \in \mathcal{L}_\Diamond$  and  $w \in W^*$ , then  $S^*vw \cdot w(\varphi) \leq v(\Diamond\varphi)$ , which yields taking join over  $w$ :

$$e^*(v, \Diamond\varphi) \leq v(\Diamond\varphi).$$

The other inequality is trivial if  $v(\Diamond\varphi) = 0$ . For the case  $v(\Diamond\varphi) > 0$ , let  $w$  be given as in the following claim then  $v(\Diamond\varphi) = \alpha = S^*vw \cdot w(\varphi) \leq e^*(v, \Diamond\varphi)$ , ending the proof of the lemma.

**Claim.** *If  $v$  is a world of  $\mathcal{M}_\Diamond$  such that  $v(\Diamond\varphi) = \alpha > 0$ , there exists a world  $w$  of  $\mathcal{M}_\Diamond$  such that  $w(\varphi) = 1$  and  $S^*vw = \alpha$ .*

*Proof:* Set

$$\Gamma_{\varphi, v} = \{\theta \in \mathcal{L}_\Diamond : v(\Diamond\theta) < \alpha\} \cup \{\neg\neg\mu : \mu \in \mathcal{L}_\Diamond, v(\Diamond\mu) = 0\}.$$

This set is not empty because  $v(\Diamond 0) = 0$  by axiom **F** $_\Diamond$ . Moreover, for any finite subset of  $\Gamma_{\varphi, v}$ , say  $\{\theta_1, \dots, \theta_n\} \cup \{\neg\neg\mu_1, \dots, \neg\neg\mu_m\}$ , we have

$$\varphi \not\vdash_{\mathcal{G}_\Diamond} \theta_1 \vee \dots \vee \theta_n \vee \neg\neg\mu_1 \vee \dots \vee \neg\neg\mu_m.$$

Otherwise, we would have

$$\begin{array}{lll} \Diamond\varphi \vdash_{\mathcal{G}_\Diamond} & \Diamond(\theta_1 \vee \dots \vee \theta_n \vee \neg\neg\mu_1 \vee \dots \vee \neg\neg\mu_m) & \mathbf{RN}_\Diamond \\ & \Diamond\theta_1 \vee \dots \vee \Diamond\theta_n \vee \Diamond\neg\neg\mu_1 \vee \dots \vee \Diamond\neg\neg\mu_m & \mathbf{D}_\Diamond \\ & \Diamond\theta_1 \vee \dots \vee \Diamond\theta_n \vee \neg\neg\Diamond\mu_1 \vee \dots \vee \neg\neg\Diamond\mu_m & \mathbf{Z}_\Diamond, \end{array}$$

which would imply by Lemma 6.2

$$v(\Diamond\varphi) \leq \max(\{v(\Diamond\theta_i) : 1 \leq i \leq n\} \cup \{v(\neg\neg\Diamond\mu_i) : 1 \leq i \leq m\}) < \alpha,$$

a contradiction. Therefore, we have by Lemma 6.1

$$T\mathcal{G}_\Diamond, \varphi \not\models \theta_1 \vee \dots \vee \theta_n \vee \neg\neg\mu_1 \vee \dots \vee \neg\neg\mu_m;$$

By Proposition 3.1 there is a Heyting algebra valuation  $u : L \rightarrow [0, 1]$  such that  $u(\varphi) = u(T\mathcal{G}_\Diamond) = 1$  and  $u(\theta) < 1$  for all  $\theta \in \Gamma_{\varphi, v}$ . Thus,  $u$  satisfies the further conditions:

- (i)  $u(\varphi) = 1$
- (ii)  $u(\theta) < 1$  if  $v(\Diamond\theta) < \alpha$ , because then  $\theta \in \Gamma_{\varphi, v}$
- (iii)  $u(\theta) = 0$  if  $v(\Diamond\theta) = 0$ , because then  $\neg\neg\theta \in \Gamma_{\varphi, v}$  and so  $u(\neg\neg\theta) < 1$  which implies  $u(\theta) = 0$ .

Let  $g : [0, 1] \rightarrow [0, 1]$  be the strictly increasing function:

$$g(x) = \begin{cases} 1 & \text{if } x = 1 \\ \delta(x+1)/2 & \text{if } 0 < x < 1 \\ 0 & \text{if } x = 0 \end{cases}$$

where  $\delta$  is given by (4). Clearly the valuation  $w = g \circ u$  inherits the properties (i), (ii) (iii) of  $u$ , with (ii) in the stronger form:

$$(ii') \quad w(\theta) < \delta \quad \text{if } v(\Diamond\theta) < \alpha$$

Moreover,  $w(\theta) > 0$  implies  $w(\theta) > \delta/2$ , by construction, and  $w(T\mathcal{G}_\Diamond) = 1$  because  $g$  is an homomorphism of Heyting algebras, hence,  $w$  belongs to  $\mathcal{M}_\Diamond$ .

To see that  $S^*vw = \alpha$ , note that  $w(\theta) \leq v(\Diamond\theta)$  whenever  $v(\Diamond\theta) < \alpha$ . If  $0 < v(\Diamond\theta)$  because then  $w(\theta) < \delta \leq v(\Diamond\theta)$  by (ii') and definition of  $\delta$ . If  $v(\Diamond\theta) = 0$  because then  $w(\theta) = 0$  by (iii). Since  $(w(\theta) \Rightarrow v(\Diamond\theta)) \geq \alpha$  for  $v(\Diamond\theta) \geq \alpha$ , and  $(w(\varphi) \Rightarrow v(\Diamond\varphi)) = (1 \Rightarrow \alpha) = \alpha$ , we have  $S^*vw = \inf_{\varphi \in \mathcal{L}_\Diamond} \{w(\varphi) \Rightarrow v(\Diamond\varphi)\} = \alpha$ . ■

**Theorem 6.1** *For any countable theory  $T$  and formula  $\varphi$  in  $\mathcal{L}_\Diamond$ ,  $T \models_{GK} \varphi$  iff  $T \vdash_{\mathcal{G}_\Diamond} \varphi$ .*

*Proof:* Assume that  $T \not\models_{\mathcal{G}_\Diamond} \varphi$ , then  $T \cup T\mathcal{G}_\Diamond \not\models \varphi$ . By strong completeness of Gödel logic, there is a Heyting algebra valuation  $v$  such that  $v(T \cup T\mathcal{G}_\Diamond) = 1$  and  $v(\varphi) < 1$ . Since  $v$  might not be a world in  $\mathcal{M}_\Diamond$  compose it with the Heyting algebra homomorphism:  $g(x) = (x+1)/2$  for  $x > 0$ ,  $g(0) = 0$ . Then

$v' = g \circ v$  belongs to  $\mathcal{M}_\Diamond$  and we still have  $v'(T) = 1$ ,  $v'(\varphi) < 1$ . Applying Lemma 6.4 to  $v'$  we have  $e^*(v', T) = 1$ ,  $e^*(v', \varphi) < 1$ . That is,  $\mathcal{M}_\Diamond \models_{v'} T$  and  $\mathcal{M}_\Diamond \not\models_{v'} \varphi$ . Hence,  $T \not\models_{\mathcal{G}_\Diamond} \varphi$ . ■

By Lemma 4.2 we have again, as in the case of  $\mathcal{L}_\Box$ , that  $\models_{GK}$  and  $\models_{GK \leq}$  coincide in  $\mathcal{L}_\Diamond$ . However,  $\models_{GK}$  no longer coincides with  $\models_{Crisp}$  as the following example illustrates.

**Example.**  $\mathcal{G}_\Diamond$  is not complete for crisp models. The formula  $\neg\neg\Diamond\varphi \rightarrow \Diamond\neg\neg\varphi$  holds in all crisp models because  $e(x, \neg\neg\Diamond\varphi) > 0$  implies that there is  $y$  such that  $Sxy \cdot e(y, \varphi) > 0$ . Thus,  $Sxy = 1$  and  $--e(y, \varphi) = 1$  showing that  $e(y, \Diamond\neg\neg\varphi) \geq Sxy \cdot (--e(y, \varphi)) = 1$ . But this formula is not a theorem of  $\mathcal{G}_\Diamond$  because it fails in the two worlds model:

$$x \xrightarrow{\frac{1}{2}} y, \quad e(x, p) = e(y, p) = 1.$$

where  $e(x, \neg\neg\Diamond p) = 1$ , and  $e(y, \Diamond\neg\neg\varphi) = \frac{1}{2}$ .

## 7 $\mathcal{G}_\Diamond$ has the finite model property

For any sentence  $\varphi$  such that  $\not\models_{\mathcal{G}_\Diamond} \varphi$  we may construct a finite counter-model inside  $\mathcal{M}_\Diamond$ .

**Theorem 7.1** *If  $\not\models_{\mathcal{G}_\Diamond} \varphi$  then there is a model  $M$  with finitely many worlds such that  $M \not\models_{GK} \varphi$ .*

*Proof:* It follows from the Claim in Lemma 6.4 that for all  $\theta$  and  $v \in \mathcal{M}_\Diamond$  there is  $w \in \mathcal{M}_\Diamond$  such that  $v(\Diamond\theta) = S^*vw \cdot w(\theta)$ . (if  $v(\Diamond\theta) = 0$  any  $w$  works). Given  $\theta$ , let  $f_\theta(v)$  be a function choosing one such  $w$  for each  $v$ . For any formula  $\theta$  let  $r(\theta)$  be the *nesting degree* of  $\Diamond$  in  $\theta$ , that is, the length of a longest chain of occurrences of  $\Diamond$  in the tree of  $\theta$ .

Given  $\varphi$  such that  $\not\models_{\mathcal{G}_\Diamond} \varphi$ , let  $v_0$  be a world (valuation) in  $\mathcal{M}_\Diamond$  such that  $v_0(\varphi) < 1$ . For each  $j \leq n = r(\varphi)$ , let  $S_j$  be the set of subformulas of  $\varphi$  of rank  $\leq j$ , and define inductively the following sets of valuations:

$$\begin{aligned} M_0 &= \{v_0\} \\ M_{i+1} &= M_i \cup \{f_\theta(v) : v \in M_i, \Diamond\theta \in S_{n-i}\} \end{aligned}$$

Clearly,  $M_n$  is finite. Consider the model induced in  $M_n$  by restricting  $e^*$  and  $S^*$  of  $\mathcal{M}_\Diamond$  to  $M_n \times Var$  and  $M_n \times M_n$  respectively. We call this model

$M_n$  for simplicity. Then for any formula  $\Diamond\theta \in S_j$  and  $v \in M_{n-j}$  there is  $w \in M_{n-(j-1)}$  such that  $v(\Diamond\theta) = S^*vw \cdot w(\theta)$ , and thus

$$v(\Diamond\theta) = \sup_w \{S^*vw \cdot w(\theta) : w \text{ is a world in } M_n\}.$$

This permits to show by induction in  $j \leq n$  that for all  $\theta \in S_j$ ,  $v \in M_{n-j}$  we have  $v(\theta) = e_{M_n}(v, \theta)$ . In particular,  $e_{M_n}(v_0, \varphi) = v_0(\varphi) < 1$ , which shows  $M_n \not\models \varphi$ . ■

The proof of the previous theorem still works if we define the accessibility relation in  $M_n$  using only subformulas of  $\varphi$ ,

$$S_n^*vw := \min_{\theta \in S_n} \{w(\theta) \Rightarrow v(\Diamond\theta)\}$$

This means that we have to use only a finite number of values of  $[0, 1]$  in the proof, and thus  $e^*$  takes values in a finite subalgebra of  $[0, 1]$ .

## 8 Modal extensions

The modal systems we have considered so far correspond to minimal modal logic, the logic of Gödel-Kripke models with an arbitrary accessibility fuzzy relation. We may consider also for each modal operator the analogues of the classical modal systems  $T$ ,  $S4$  and  $S5$ , usually presented as combinations of the following axioms:

$$\begin{array}{ll} \mathbf{T}_\Box: & \Box\varphi \rightarrow \varphi \\ \mathbf{4}_\Box: & \Box\varphi \rightarrow \Box\Box\varphi \\ \mathbf{B}_\Box: & \varphi \rightarrow \Box\neg\Box\neg\varphi \end{array} \quad \begin{array}{ll} \mathbf{T}_\Diamond: & \varphi \rightarrow \Diamond\varphi \\ \mathbf{4}_\Diamond: & \Diamond\Diamond\varphi \rightarrow \Diamond\varphi \\ \mathbf{B}_\Diamond: & \varphi \rightarrow \neg\Diamond\neg\Diamond\varphi \end{array}$$

Call a GK-model  $\mathcal{M} = \langle W, S, e \rangle$  *reflexive* if  $Sxx = 1$  for all  $x \in W$ , *(min)transitive* if  $Sxy \cdot Syz \leq Sxz$  for all  $x, y, z$ , and *symmetric* if  $Sxy = Syx$  for all  $x, y \in W$ .

**Proposition 8.1**  $\mathbf{T}_\Box$  and  $\mathbf{T}_\Diamond$  are valid in all reflexive GK-models,  $\mathbf{4}_\Box$  and  $\mathbf{4}_\Diamond$  are valid in all transitive GK-models,  $\mathbf{B}_\Box$  and  $\mathbf{B}_\Diamond$  are valid in all GK-symmetric models.

*Proof:* If  $Sxx = 1$  for all  $x$  then  $(\mathbf{T}_\Box)$ :  $e(x, \Box\varphi) \leq (Sxx \Rightarrow e(x, \varphi)) = e(x, \varphi)$ , and  $(\mathbf{T}_\Diamond)$ :  $e(x, \Diamond\varphi) \geq Sxx \cdot e(x, \varphi) = e(x, \varphi)$ .

Assume  $Sxy \cdot Syz \leq Sxz$  for all  $x, y, z$ . ( $\mathbf{4}_\square$ ):  $e(x, \square\varphi) \cdot Sxy \cdot Syz \leq (Sxz \Rightarrow e(z, \varphi)) \cdot Sxz \leq e(z, \varphi)$ . Hence,  $e(x, \square\varphi) \cdot Sxy \leq (Syz \Rightarrow e(z, \varphi))$ . Taking meet over  $z$  in the right hand side:  $e(x, \square\varphi) \cdot Sxy \leq e(y, \square\varphi)$ ; hence,  $e(x, \square\varphi) \leq (Sxy \Rightarrow e(y, \square\varphi))$  for all  $y$  and thus  $e(x, \square\varphi) \leq e(x, \square\square\varphi)$ . ( $\mathbf{4}_\diamond$ ): For any  $x, y, z$ ,  $Sxy \cdot Syz \cdot e(z, \varphi) \leq Sxz \cdot e(z, \varphi) \leq e(x, \diamond\varphi)$ . Hence,  $Syz \cdot e(z, \varphi) \leq (Sxy \Rightarrow e(x, \diamond\varphi))$ . Taking join over  $z$  in the left,  $e(x, \diamond\varphi) \leq (Sxy \Rightarrow e(x, \diamond\varphi))$ , thus  $Sxy \cdot e(x, \diamond\varphi) \leq e(x, \diamond\varphi)$ . Taking join again in the left,  $e(x, \diamond\diamond\varphi) \leq e(x, \diamond\varphi)$ .

Assume  $Sxy = Syx$  for all  $x, y$ . ( $\mathbf{B}_\square$ ): We prove the stronger  $\neg\varphi \rightarrow \square\neg\square\varphi$ . Assume  $e(x, \neg\varphi) > 0$  then  $e(x, \varphi) = 0$ . Take any  $y$  such that  $Sxy > 0$ , then  $e(y, \square\varphi) \leq (Syx \Rightarrow e(x, \varphi)) = (Sxy \Rightarrow e(x, \varphi)) = 0$ . Therefore,  $e(y, \neg\square\varphi) = 1$ , and  $(Sxy \Rightarrow e(y, \neg\square\varphi)) = 1$ . This shows that  $x(\square\neg\square\varphi) = 1$ . ( $\mathbf{B}_\diamond$ ): Suppose  $e(x, \varphi) > e(x, \neg\diamond\neg\diamond\varphi)$  then  $e(x, \neg\diamond\neg\diamond\varphi) = 0$  and  $e(x, \diamond\neg\diamond\varphi) = 1$ . This means that there is  $y$  such that  $Sxy \cdot e(x, \neg\diamond\varphi) > 0$  thus  $Sxy > 0$  and  $e(x, \neg\diamond\varphi) = 1$ , hence  $e(y, \diamond\varphi) = 0$ , therefore,  $Syx \cdot e(x, \varphi) = 0$  which is absurd because  $Syx = Sxy > 0$  and  $e(x, \varphi) > 0$  by construction. ■

Let *Ref*, *Trans*, and *Symm* denote the GK-classes of models satisfying, respectively, reflexivity, transitivity, and symmetry, and let  $\models_{\mathcal{C}}$  denote semantic consequence with respect to models in the class  $\mathcal{C}$ .

**Theorem 8.1** (i)  $\mathcal{G}_\square + \mathbf{T}_\square$  and  $\mathcal{G}_\diamond + \mathbf{T}_\diamond$  are strongly complete (for countable theories) with respect to  $\models_{Ref}$ .  
(ii)  $\mathcal{G}_\square + \mathbf{4}_\square$  and  $\mathcal{G}_\diamond + \mathbf{4}_\diamond$  are strongly complete with respect to  $\models_{Trans}$ .  
(iii)  $\mathcal{GS4}_\square := \mathcal{G}_\square + \mathbf{T}_\square + \mathbf{4}_\square$  and  $\mathcal{GS4}_\diamond := \mathcal{G}_\diamond + \mathbf{T}_\diamond + \mathbf{4}_\diamond$  are strongly complete with respect to  $\models_{Ref \cap Trans}$ .

*Proof:* Soundness follows from Proposition 8.1. Completeness follows, in each case, by asking the worlds of the canonical models introduced in the completeness proofs of  $\mathcal{G}_\square$  and  $\mathcal{G}_\diamond$  to satisfy the corresponding schemes. The key fact is that these schemes force the accessibility relations  $S_\square^*vw = \inf_{\varphi \in \mathcal{L}_\square} \{v(\square\varphi) \Rightarrow w(\varphi)\}$  and  $S_\diamond^*vw = \inf_{\varphi \in \mathcal{L}_\diamond} \{v(\varphi) \Rightarrow w(\diamond\varphi)\}$  to satisfy the respective properties. (i) If  $v(\mathbf{T}_\square) = 1$  then  $S_\square^*vv = \inf_{\varphi \in \mathcal{L}_\square} \{v(\square\varphi \rightarrow \varphi)\} = 1$ . If  $v(\mathbf{T}_\diamond) = 1$  then  $S_\diamond^*vv = \inf_{\varphi \in \mathcal{L}_\diamond} \{v(\varphi \rightarrow \diamond\varphi)\} = 1$ . (ii) If  $v(\mathbf{4}_\square) = 1$  then  $v(\square\varphi) \leq v(\square\square\varphi)$  and so

$$\begin{aligned} S_\square^*vv' \cdot S_\square^*v'v'' &\leq [(v(\square\square\varphi) \Rightarrow v'(\square\varphi)) \cdot (v'(\square\varphi) \Rightarrow v''(\varphi))] \\ &\leq (v(\square\square\varphi) \Rightarrow v''(\varphi)) \leq (v(\square\varphi) \Rightarrow v''(\varphi)) \end{aligned}$$

Taking meet over  $\varphi$  in the last formula we get:  $S_\square^*vv' \cdot S_\square^*v'v'' \leq S_\square^*vv''$ . (iii)



If  $v(\mathbf{4}_\diamond) = 1$  then  $v(\diamond\diamond\varphi) \leq v(\diamond\varphi)$  and thus

$$\begin{aligned} S_\diamond^* vv' \cdot S_\diamond^* v'v'' &\leq [(v'(\diamond\varphi) \Rightarrow v(\diamond\diamond\varphi)) \cdot (v''(\varphi) \Rightarrow v'(\diamond\varphi))] \\ &\leq (v''(\varphi) \Rightarrow v(\diamond\diamond\varphi)) \leq (v''(\varphi) \Rightarrow v(\diamond\varphi)) \end{aligned}$$

Minimizing over  $\varphi$  in the last formula we get  $S_\diamond^* vv' \cdot S_\diamond^* v'v'' \leq S_\diamond^* vv''$ . ■

One of the original motivations of the second author to study these fuzzy modal logics was to interpret the possibility operator  $\diamond$  in the class of Gödel frames  $Ref\ell \cap Trans \cap Symm$  as a notion of similarity in the sense of Godo and Rodríguez [9], and a reasonable conjecture was that  $\mathcal{GS5}_\diamond = \mathcal{GS4}_\diamond + \mathbf{B}_\diamond$  would axiomatize validity in this frames. Unfortunately, the axioms  $\mathbf{B}_\square$ ,  $\mathbf{B}_\diamond$  do not force symmetry in the canonical models. Thus, we have not been able to show completeness of  $\mathcal{G}_\square + \mathbf{B}_\square$  or  $\mathcal{G}_\diamond + \mathbf{B}_\diamond$  for  $\models_{Symm}$ , even the less completeness of  $\mathcal{GS5}_\diamond$  or  $\mathcal{GS5}_\square = \mathcal{GS4}_\square + \mathbf{B}_\square$  with respect to  $\models_{Ref\ell \cap Trans \cap Symm}$ . Perhaps stronger symmetry axioms as

$$\begin{aligned} (\varphi \rightarrow \square\theta) &\rightarrow \square(\square\varphi \rightarrow \theta) \\ \diamond(\diamond\varphi \rightarrow \theta) &\rightarrow (\varphi \rightarrow \diamond\theta) \end{aligned}$$

would do. In any case, it is possible to show that validity in Gödel  $Ref\ell \cap Trans \cap Symm$  is decidable.

## 9 Adding truth constants

The previous results on strong completeness may be generalized to languages with a set  $Q \subseteq [0, 1]$  of truth values added as logical constants to the language, provided  $Q$  is topologically discrete and well-ordered, in particular when  $Q$  is finite.

Introduce a constant connective symbol for each  $r \in Q$ , denoted by  $r$  itself excepting 0 and 1 which are identified with  $\perp$  and  $\top$ . Let  $\mathcal{G}_\square(Q)$  be the logic obtained by adding to  $\mathcal{G}_\square$  the axiom schemes R1 - R4 below, and let  $\mathcal{G}_\diamond(Q)$  be defined similarly by adding to  $\mathcal{G}_\diamond$  the book-keeping axioms R1 and R5 - R7, for all  $r, s \in Q$ :

- R1. (book-keeping axioms)
  - $r \rightarrow s$ , if  $r \leq s$ ,
  - $(r \rightarrow s) \rightarrow s$ , if  $s < r$
- R2.  $r \rightarrow \square r$
- R3.  $(r \rightarrow \square\theta) \rightarrow \square(r \rightarrow \theta)$
- R4.  $((\square\theta \rightarrow r) \rightarrow r) \rightarrow \square((\theta \rightarrow r) \rightarrow r)$
- R5.  $\diamond r \rightarrow r$

- R6.  $\Diamond(r \rightarrow \varphi) \rightarrow (r \rightarrow \Diamond\varphi)$   
R7.  $\Diamond((\varphi \rightarrow r) \rightarrow r) \rightarrow ((\Diamond\varphi \rightarrow r) \rightarrow r)$

Note that the double negation shift axioms  $\mathbf{Z}_\square$  and  $\mathbf{Z}_\Diamond$  become superfluous (follow from R4 and R7, respectively). Moreover, R2+R3 may be replaced by the single axiom:  $(r \rightarrow \square\theta) \leftrightarrow \square(r \rightarrow \theta)$ , and R5+R6 by  $\Diamond(r \rightarrow \varphi) \leftrightarrow (r \rightarrow \Diamond\varphi)$ .

The evaluation of GK-models is extended by defining:  $e(x, r) = r$  for each  $r \in Q$ . It may be shown then that  $\mathcal{G}_\square(Q)$  and  $\mathcal{G}_\Diamond(Q)$  are strongly complete for countable theories in their respective languages, and the same holds for the logics mentioned in Theorem 8.1.

This extends substantially a result of Esteva, Godo and Nogera [4] on weak completeness of Gödel logic with rational truth constants. If one is interested in weak completeness only, no condition is needed on  $Q$  since it is enough to consider the finitely many truth constants appearing in the sentence to be proved. On the other hand, discreteness of  $Q$  is necessary for strong completeness: if  $r$  is a limit points in  $Q$  then there is a strictly increasing or decreasing sequence converging to  $r$ , say  $\{r_n\}$  increases to  $\sup r_n = r$ , then

$$\{r_1 \rightarrow \theta, r_2 \rightarrow \theta, r_3 \rightarrow \theta, \dots\} \models_{GK} r \rightarrow \theta$$

but no finite subset of premises can grants this, thus by soundness

$$\{r_1 \rightarrow \theta, r_2 \rightarrow \theta, r_3 \rightarrow \theta, \dots\} \not\models_{\mathcal{G}_\square(Q)} r \rightarrow \theta,$$

Discreteness is not enough, however:  $Q = \{r_1 < r_2 < \dots \quad \dots < q_2 < q_1\}$  with  $\sup r_i = \inf q_i$  is discrete, and

$$r_1 \rightarrow \theta, r_2 \rightarrow \theta, \dots, \psi \rightarrow q_1, \psi \rightarrow q_2, \dots \models_{GK} \psi \rightarrow \theta$$

but no finite subset of the premises yields the same consequence.

## 10 Comment

It rests to axiomatize validity and consequence of the full logic with both modal operators combined. It may be seen that the union of the systems  $\mathcal{G}_\square$  and  $\mathcal{G}_\Diamond$  is not enough for that purpose. However,  $\mathcal{G}_\square \cup \mathcal{G}_\Diamond$  together with Fischer Servi [5] "connecting axioms":

$$\begin{aligned} \Diamond(\varphi \rightarrow \psi) &\rightarrow (\square\varphi \rightarrow \Diamond\psi) \\ (\Diamond\varphi \rightarrow \square\psi) &\rightarrow \square(\varphi \rightarrow \psi) \end{aligned}$$

may be proved to be a strongly complete axiomatization. This will be shown in a sequel of this paper.

## References

- [1] Mathias Baaz and Richard Zach, Compact Propositional Logics. Proc. International Symp on multiple valued logic, IEEE Computer Society Press, 1998, pp 108-113.
- [2] A. Chagrov and M. Zakharyashev, Modal Logic, Clarendon Press, Oxford, 1997.
- [3] Milan Božić and Kosta Došen, Models for Normal Intuitionistic Modal Logics. *Studia Logica* 43, 3 (1984) 217-245.
- [4] Francesc Esteva, Lluís Godo, Carles Noguera, On rational Gödel and nilpotent minimum logics. Tenth international conference IPMU 2004. Information processing and management of uncertainty in knowledge-based systems. Vol. I, Casa editrice Università La Sapienza, pp. 561-568.
- [5] G. Fisher Servi. Axiomatizations for some intuitionistic modal logics, *Rend. Sem. Mat. Polit de Torino* 42 (1984) 179-194.
- [6] Melving Fitting. Many valued modal logics. *Fundamenta Informaticae* 15 (1991) 325-254.
- [7] Melving Fitting. Many valued modal logics, II. *Fundamenta Informaticae* 17 (1992) 55-73.
- [8] Josep M. Font, Modality and Possibility in some intuitionistic modal logics. *Notre Dame Journal of Formal Logic* 27, 4 (1986) 533-546
- [9] Lluís Godo and Ricardo Rodríguez. A fuzzy modal logic for similarity reasoning. In Guoqing Chen and MinGMheng Ying and Kai-Yuan Cai, editors, *Fuzzy Logic and Soft Computing*. Kluwer Academic, 1999.
- [10] C. Grefe, Fisher Servi's intuitionistic modal logic has the finite model property, In *Advances in Modal Logic Vol.1* CSLI, Stanford, 1998.
- [11] Peter Hájek, Metamathematics of fuzzy logic, in: *Trends in Logic*, Vol. 4, Kluwer Academic Publishers, Dordrecht, 1998.
- [12] H. Ono. On some intuitionistic modal logics. *Publications of the Research Institute for Mathematical Sciences, Kyoto University* 13 (1977) 13-55.
- [13] Frank Wolter, Superintuitionistic Companions of Classical Modal Logics. *Studia Logica* 58, 3 (1997) 229-295.